Answer sheet 8

Assignment 1. (a) $(AB)_{ik} = \sum_{j=1}^m a_{ij}b_{jk}$ thus

$$\operatorname{tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji} = \sum_{j=1}^n \sum_{i=1}^n b_{ji}a_{ij} = \operatorname{tr}(BA).$$

(b) This follows from (a) with $A' = A$ and $B' = BC$.
(c) By linearity of the expected value, $\mathbb{E}(\operatorname{tr}(A)) = \mathbb{E}\sum_{i=1}^n a_{ii} = \sum_{i=1}^n \mathbb{E}(a_{ii}) = \operatorname{tr}(\mathbb{E}(A))$.

Assignment 2. Let $v \in \mathbb{R}^p \setminus \{0\}$ such that $Qv = \lambda v$. Then

$$\lambda v = Qv = QQv = Q\lambda v = \lambda Qv = \lambda^2 v.$$

As $v \neq 0$ this implies $\lambda = \lambda^2$; equivalently $\lambda \in \{0, 1\}$.

Assignment 3. (a) There exists $u \in \mathbb{R}^p$ such that $v = Pu = PPu = Pv$.
(b) We have $(Pw)^T x = w^T P^T x = w^T (Px) = 0$ because $w \in W$ must be orthogonal to $Px \in V$. This means that Pw is orthogonal to everything and hence equals 0.
(c) Each $x \in \mathbb{R}^p$ can be written (uniquely) as $v + w$, $v \in V$, $w \in V^\perp$. Since P and Q agree on V and $V^\perp$, they must agree throughout $\mathbb{R}^p$.

Assignment 4. Notice that $P\binom{x}{y} = \binom{x}{x}$ and $Q\binom{x}{y} = \binom{y}{y}$. Thus $M(P) = M(Q) = span((1,1))$. Furthermore $P^2\binom{x}{y} = P\binom{x}{x} = \binom{x}{x} = P\binom{x}{y}$, so $P^2 = P$, and similarly $Q^2 = Q$.

Assignment 5. (a) For each $u = (u_1, \dots, u_p) \in \mathbb{R}^p$ we have $Xu = u_1x_1 + \cdots + u_px_p$, and these constitute precisely the elements of V.
(b) If $X^TXv = 0$, then

$$\|Xv\|^2 = v^TX^TXv = 0,$$

which means that $Xv = 0$. By part (a), Xv is a linear combination of the columns of X. Since these are independent, it must be that $v = 0$. As the $p \times p$ matrix X^TX is injective, it must be invertible.
(c) To see that X is a projection simply note that

$$H^2 = X(X^TX)^{-1}X^TX(X^TX)^{-1}X^T = X(X^TX)^{-1}X^T = H,$$

and

$$H^T = (X(X^TX)^{-1}X^T)^T = (X^T)^T[(X^TX)^{-1}]^TX^T = X([X^TX]^T)^{-1}X^T = X(X^TX)^{-1}X^T = H.$$

Clearly $Hy = X[(X^TX)^{-1}X^Ty] \in V$, and so $M(H) \subseteq V$. Conversely, if $y \in V$ then $y = Xu$ for some $u \in \mathbb{R}^p$ and then $Hy = HXu = Xu = y$, so $y \in M(H)$. This completes the proof.

Assignment 6. (a) Otherwise, we can remove a subset of them without changing the span, and do so repeatedly until we have an independent set.
(b) This is so because Hy must belong to the column space of X, hence equal Xv for some v. Since everything is linear v should be a linear function X, $v = My$, and then $H = XM$.
(c) For any $y \in V^\perp$, $Hy = 0$, which means that X_i^Ty has to be zero. These are precisely the coordinates of the p-dimensional vector X^Ty, which then should be zero. Conversely, if

$y \notin V^{\perp}$, then $X_i^T y$ will be nonzero for some i, and so $X^T y$ will not be zero. Thus X^T is the "minimal" matrix with kernel $V^{\perp}$.
(d) We know that $Hx_i = x_i$ for all i, and using the hint

$$Xe_i = x_i = Hx_i = XBX^T x_i = XBX^T Xe_i.$$

Since X is injective, this means that $BX^T Xe_i = e_i$. This holds for all i, which means that $BX^T X$ is the identity and then $B = (X^T X)^{-1}$.

Assignment 7. Let $\Omega = U\Lambda U^T$ be the spectral decomposition of Ω, and let $\lambda_i = \Lambda_{ii}$ be the eigenvalues of Ω (in an arbitrary order). Then for any $v \in \mathbb{R}^p$ we have

$$v^T \Omega v = \sum_{i=1}^{p} [Uv]_i^2 \lambda_i.$$

If all the λ_i's are (strictly) positive, then this is (strictly) positive for all $v \neq 0$ (because U is injective, so $Uv \neq 0$). If one $\lambda_i < 0$ then choosing $[Uv]_j$ to be 0 for $j \neq i$ and 1 for $j = i$ gives $v^T \Omega v < 0$. Such a choice is possible since U is surjective.

Assignment 8. Clearly such Q is symmetric, and by orthonormality

$$Q^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} v_i v_i^T v_j v_j^T = \sum_{i=1}^{k} v_i v_i^T v_i v_i^T = \sum_{i=1}^{k} v_i v_i^T = Q.$$

Since $Qv_i = v_i$ for all i and $Qv = 0$ for all $v \in [span(v_1, \ldots, v_k)]^{\perp}$, Q is the projection on this span and hence of rank k.
Conversely, if Q is a projection, we can let $v_1, \ldots, v_k$ be an orthonormal basis of $M(Q)$. Let V be a matrix with columns $v_1, \ldots, v_k$. Then we know that $Q = V(V^T V)^{-1} V^T = VV^T$, and it remains to show that this is the same matrix as $Q' = \sum_{i=1}^{k} v_i v_i^T$. Since $v_j = Ve_j$ for the unit vector e_j and the V_i's are orthogonal,

$$Qv_j = VV^T Ve_j = Ve_j v_j = \sum_{i=1}^{k} v_i v_i^T v_j = Q' v_j.$$

If $v^T v_j = 0$ for all j, then clearly $Qv = 0 = Q'v$. The proof is achieved.

Assignment 9. (a) If U is orthogonal, then $W = UZ \sim N(0, UIU^T) = N(0, I)$. Let $H = U\Lambda U^T$ be a spectral decomposition of H with the first r elements of Λ equal to one and the rest equal to zero (in view of a previous assignment). Then

$$Z^T HZ = W^T \Lambda W = \sum_{i=1}^{r} W_i^2 \sim \chi_r^2.$$

(We used the fact that the marginal law of $(W_1, \ldots, W_r)$ is $N(0_r, I_{r\times r})$.
(b) Define $Z = \Omega^{-1/2}(Y - \mu) \sim N(0, \Omega^{-1/2}\Omega\Omega^{-1/2}) = N(0_p, I_{p\times p})$. Then

$$(Y - \mu)^T \Omega^{-1} (Y - \mu) = Z^T Z \sim \chi_p^2.$$