Answer sheet 6

Assignment 1. This assignment is postponed to next week.

Assignment 2. (a) For a given level α, we reject the null hypothesis if $\sum X_i \leq Gamma_{n,4,\alpha}$, the α-quantile of the $Gamma(n, 4)$ distribution. The p-value is the smallest α for which we reject, which is the smallest α such that

$$Gamma_{16,4,\alpha} \geq 4.$$

We obtain a plot of $Gamma_{16,4,\alpha}$ as a function of α using the following code :

```
n <- 16
f <- function(alpha) qgamma(alpha, shape = n, rate = 4)
curve(f, from = 0, to = 1)
sample.sum <- 4
abline(h = sample.sum)
```

A graphical inspection reveals that the p-value is approximately 0.533.
(b) After the code in (a), use the following commands :

```
p <- uniroot(function(alpha) f(alpha) - sample.sum, interval = c(0.0001, 1))$root
abline(v = p)
```

(c) Here the p-value is 0.012.
(d) We reject if and only if the p-value is smaller than $\alpha = 0.05$, which happens in the second case but not the in the first.
(e) Use `qgamma(0.05, shape = 16, rate = 4)` to obtain 2.508989.

Assignment 3. (a) We have $\overline{X} \sim N(\mu, 1/n)$.
(b) Under H_0, $\sqrt{n}\overline{X} \sim N(0, 1)$. Letting Φ denote the Gaussian distribution function, we obtain the equation

$$1 - \alpha = \mathbb{P}(-v_\alpha \leq \overline{X} \leq v_\alpha) = \mathbb{P}(-\sqrt{n}v_\alpha \leq \sqrt{n}\overline{X} \leq \sqrt{n}v_\alpha) = \Phi(\sqrt{n}v_\alpha) - \Phi(-\sqrt{n}v_\alpha).$$

By symmetry the right hand-side equals $2\Phi(\sqrt{n}v_\alpha) - 1$. Thus $1 - \alpha/2 = \Phi(\sqrt{n}v_\alpha)$ and $v_\alpha = n^{-1/2}\Phi^{-1}(1 - \alpha/2) = n^{-1/2}z_{1-\alpha/2}$.
(c) The p-value is the infimum of the α's for which we reject,

$$p = p(X_1, \ldots, X_n) = \inf\{\alpha : |\overline{X}| > v_\alpha\} = \inf\{\alpha : |\overline{X}| > n^{-1/2}z_{1-\alpha/2}\}.$$

Since $z_{1-\alpha/2}$ is continuous and decreasing in α, the infimum is attained when $|\overline{X}|$ equals the threshold

$$|\overline{X}| = n^{-1/2}z_{1-p/2} = n^{-1/2}\Phi^{-1}(1 - p/2) \quad \Longrightarrow \quad p(X_1, \ldots, X_n) = 2(1 - \Phi(\sqrt{n}|\overline{X}|)).$$

(d) The code below carries out the simulation :

```
set.seed(25102017)
mu <- 0
n <- 11
REP <- 1000
p <- numeric(REP)
for(i in 1:REP)
{
  X <- rnorm(n, mean = mu, sd = 1)
  p[i] <- 2 - 2 * pnorm(sqrt(n) * abs(mean(X)))
}
hist(p)
```

The resulting histrogram suggests that p-value is uniformly distributed when H_0 holds ; in fact this can be shown to hold true in the continuous case by means of the probability transform. If μ is different than zero, than the histogram is concentrated around zero ; the concentration increases with n and with $|\mu|$.

Assignment 4. (a) We know that $\sqrt{n}(\widehat{\theta}_n - \theta) \to N(0, 1/I_1(\theta))$ (slide 161). The asymptotic variance is therefore $v(\theta) = 1/(nI_1(\theta))$.
(b) We have

$$T = nI_1(\widehat{\theta})(\widehat{\theta} - \theta_0)^2.$$

(c) Since v is continuous $v(\widehat{\theta})/v(\theta) \to 1$ in probability. By Slutsky's theorem

$$T = nI_1(\theta)(\widehat{\theta} - \theta_0)^2 \frac{v(\theta)}{v(\widehat{\theta})} = \left(\sqrt{nI_1(\theta)}(\widehat{\theta} - \theta_0)^2\right) \frac{v(\theta)}{v(\widehat{\theta})} \to \chi_1^2.$$

(d) Write $\theta = \sigma^2$ to avoid differentiation errors. The log likelihood and its derivatives are

$$\begin{aligned}
\ell(x_1, \ldots, x_n; \theta) &= -\frac{n}{2}\ln(2\pi\theta) - \frac{\sum_{i=1}^n x_i^2}{2\theta} \\
\ell'(x_1, \ldots, x_n; \theta) &= \frac{\sum x_i^2}{2\theta^2} - \frac{n}{2\theta} \quad \Longrightarrow \quad \widehat{\theta} = \frac{1}{n}\sum_{i=1}^n x_i^2. \\
\ell''(x_1, \ldots, x_n; \theta) &= \frac{n}{2\theta^2} - \frac{\sum x_i^2}{\theta^3} \quad \Longrightarrow \quad \ell''(\widehat{\theta}) = -\frac{n}{2\widehat{\theta}^2} < 0,
\end{aligned}$$

so $\widehat{\theta}$ is a maximizer and $nI_1(\widehat{\theta}) = I_n(\widehat{\theta}) = -\mathbb{E}\ell''(\widehat{\theta}) = n/2\widehat{\theta}^2$. We obtain the Wald test statistic

$$T = \frac{n}{2\widehat{\theta}^2}(\widehat{\theta} - \theta_0)^2 = \frac{n}{2}\left(1 - \frac{\sigma_0^2}{\widehat{\sigma}^2}\right)^2, \qquad \widehat{\sigma}^2 = \widehat{\theta}.$$

Since T is asymptotically χ_1^2, the approximate Wald test rejects H_0 if T is larger than the $(1-\alpha)$-quantile of the χ_1^2 distribution, $\chi^2_{1,1-\alpha}$.
Remark. The distribution of $\sum x_i^2/\sigma_0^2$ is χ_n^2, so we can get an exact Wald test, but it will not have an explicit form.
(e) The likelihood ratio is

$$\Lambda = \left(\frac{\sigma_0^2}{\widehat{\sigma}^2}\right)^{n/2} \exp\left(\frac{n}{2}\frac{\widehat{\sigma}^2}{\sigma_0^2}\right) \exp\left(-\frac{n}{2}\right).$$

and twice its logarithm is asymptotically χ_1^2. The asymptotic test rejects therefore when

$$n\left[\frac{\widehat{\sigma}^2}{\sigma_0^2} - \log\frac{\widehat{\sigma}^2}{\sigma_0^2} - 1\right] > \chi^2_{1,1-\alpha}.$$

The tests are not the same, but can be shown (by a Taylor expansion, essentially) to be rather close to each other.

Assignment 5. (a) (i)The likelihood ratio test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is given by $\mathbf{1}(\sqrt{n}|\overline{X} - \theta_0| > q_{1-\alpha/2})$. So, the associated $100(1-\alpha)\%$ confidence interval for θ is given by

$$\{\theta : \sqrt{n}|\overline{X} - \theta| \leq q_{1-\alpha/2}\} = \left[\overline{X} - q_{1-\alpha/2}/\sqrt{n},\ \overline{X} + q_{1-\alpha/2}/\sqrt{n}\right].$$

(ii) In this case $\mathcal{I}_n(\theta) = n$, which is free of the parameter θ. Hence, the Wald test is given by $\mathbf{1}(n(\overline{X} - \theta_0)^2 > \chi^2_{1,1-\alpha})$. So, the associated $100(1-\alpha)\%$ confidence interval for θ is given by

$$\{\theta : n(\overline{X} - \theta)^2 \leq \chi^2_{1,1-\alpha}\} = \left[\overline{X} - \sqrt{\chi^2_{1,1-\alpha}/n},\ \overline{X} + \sqrt{\chi^2_{1,1-\alpha}/n}\right].$$

However, observe that $\sqrt{\chi^2_{1,1-\alpha}} = q_{1-\alpha/2}$ because of the following reason. Let $Z \sim N(0,1)$ and c be such that $P(|Z| > c) = \alpha \Leftrightarrow c = q_{1-\alpha/2} > 0$ since $\alpha < 1$. However, this is the same as saying $P(Z^2 > c^2) = \alpha \Leftrightarrow c^2 = \chi^2_{1,1-\alpha}$. So, $\sqrt{\chi^2_{1,1-\alpha}} = q_{1-\alpha/2}$. Thus, the two confidence intervals are the same.

(b) (i) For testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$, Wilks' theorem applied to the likelihood ratio statistic yields the test $\mathbf{1}(2n\overline{X}\log(\overline{X}/p_0) + 2(n - n\overline{X})\log\{(1-\overline{X})/(1-p_0)\} > \chi^2_{1,1-\alpha})$. So, the associated $100(1-\alpha)\%$ confidence interval for p is given by

$$\{p : 2n\overline{X}\log(\overline{X}/p) + 2(n - n\overline{X})\log\{(1-\overline{X})/(1-p)\} \leq \chi^2_{1,1-\alpha}\}.$$

Observe that the above inequation does not in general yield a closed form expression of the confidence interval.

(ii) In this case $\mathcal{I}_n(p) = n/(p(1-p))$. Hence, the Wald test is given by $\mathbf{1}(n(\overline{X} - p_0)^2/(\overline{X}(1 - \overline{X})) > \chi^2_{1,1-\alpha})$. So, the associated $100(1-\alpha)\%$ confidence interval for p is given by

$$\{p : n(\overline{X} - p)^2/(\overline{X}(1-\overline{X})) \leq \chi^2_{1,1-\alpha}\}.$$

The above inequation can be solved explicitly (being a quadratic) to obtain a confidence interval for p.

(iii) The asymptotic test using the convergence in distribution as given is $\mathbf{1}(\sqrt{n}|\overline{X} - p_0| > q_{1-\alpha/2}\sqrt{p_0(1-p_0)})$. So, the associated $100(1-\alpha)\%$ confidence interval for p is given by

$$\{p : \sqrt{n}|\overline{X} - p| \leq q_{1-\alpha/2}\sqrt{p(1-p)}\} = \{p : n(\overline{X} - p)^2 \leq q^2_{1-\alpha/2}p(1-p)\}.$$

Once again, the above inequation can be solved explicitly (being a quadratic) to obtain a confidence interval for p.

(iv) No, these confidence intervals are not the same.

We can compare their lengths in the following way : set $\overline{x} = 0.5$ and $\alpha = 0.05$. Then, the lengths of the three confidence intervals are found to be

(1) when $n = 10$, (2) when $n = 25$, and (3) when $n = 100$.

Assignment 6. This assignment is postponed to next week.

Assignment 7. (i). The model

$X_i \sim \mathrm{N}(\mu, \sigma^2)$	for every $i \in \{1, \dots, 12\}$,
$X_1, \dots, X_{12}$	independent,
the parameters μ and σ^2	unknown.

The null and the alternative hypothesis are :

$$H_0 : \mu = 12.2, \quad H_1 : \mu \neq 12.2.$$

(ii). As seen in class, you can pick the statistics

$$T = \sqrt{n}\,\frac{\bar{X}_n - \mu_0}{S_n},$$

where μ_0 is the value under H_0, here $\mu_0 = 12.2$.
We can see that T is "small" if H_0 is true, and "large" is H_1 is true. We note as well that $\bar{X}_n$ is an estimator of the true value of μ. So if H_0 is true we expect that $\bar{X}_n \approx \mu_0$ and $T \approx 0$. On the other hand if H_1 is true we expect that $\bar{X}_n \approx \mu \neq \mu_0$ and $T >> 0$ or $T << 0$.
We could also consider $|T|$ as a test statistics and expect small values under H_0 and large under H_1.

(iii). Extreme values correspond to a very large $|T|$, that is for $|T| > c$, where c is a critical value.
To find c remember that we want the probability of the type I error (reject H_0 when it's true) to be equal to α. In our case

$$\alpha = \mathbb{P}_{H_0}(\{T < -c\} \cup \{T > c\}) = 1 - \mathbb{P}_{\mu=\mu_0}(-c \leq T \leq c). \tag{1}$$

We know that (slide 207)

$$\sqrt{n}\,\frac{\bar{X}_n - \mu}{S_n} \sim t_{n-1}.$$

If H_0 is true, $\mu = \mu_0$, so $T \sim t_{n-1}$. Hence to satisfy the condition of (1) we can take $c = t_{n-1}(1 - \alpha/2)$.

(iv). $\alpha = 0.05$, so we reject H_0 in favour of H_1 if

$$\left|\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n}\right| > t_{11}(0.975).$$

$$\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n} = 2.002 \quad \text{and } t_{11}(0.975) = 2.20,$$

and we do not have enough evidence to reject H_0.
(Which doesn't mean that we "accept" H_0 !).

(v). For $\alpha = 0.10$ we reject H_0 in favour of H_1 if

$$\left|\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n}\right| > t_{11}(0.95).$$

$$\sqrt{12}\,\frac{\bar{X}_n - 12.2}{S_n} = 2.002 \quad \text{and } t_{11}(0.975) = 1.80,$$

and this time we do reject H_0.
The difference w.r.t. part ((iv)) is that if we allow a bigger type I error we are satisfied with less evidence to make a decision against H_0.

(vi).

$$p_{obs} = \mathbb{P}_{H_0}(\{T < -2.002\} \cup \{T > 2.002\}) = 1 - \mathbb{P}_{\mu=\mu_0}(-2.002 \leq T \leq 2.002).$$

If H_0 is true $T \sim t_{11}$, so

$$p_{obs} = 1 - \left(F_{t_{11}}(2.002) - F_{t_{11}}(-2.002)\right),$$

where $F_{t_{11}}$ is the cdf of the t_{11} law. Exploiting the symmetry of this distribution around 0 we obtain that

$$p_{obs} = 2\left(1 - F_{t_{11}}(2.002)\right) = 2(1 - 0.9647) = 0.071.$$

(vii). $p_{obs} > 0.05$, so we do not reject H_0 in favour of H_1 at a 5 % level, while $p_{obs} < 0.10$, thus we do reject H_0 at a 10 % significance level.
We could say that p_{obs} is the smallest level for which we would rejectH_0 in favour of H_1.

Assignment 8. This assignment is postponed for next week. Meanwhile, you can check this out : https ://xkcd.com/882