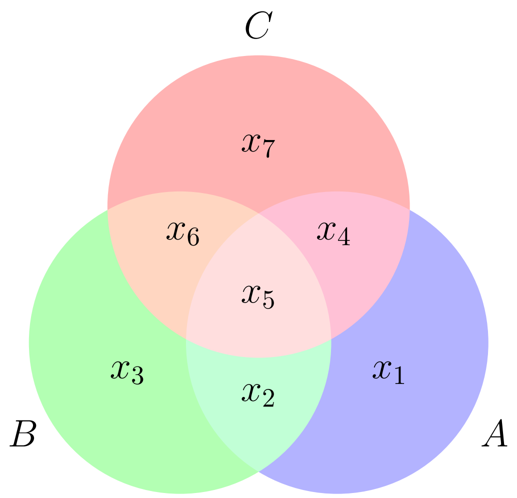
Answer sheet 1

Assignment 1. Define $x_1 = P(A \cap (B \cup C)^c)$, $x_2 = P(A \cap B \cap C^c)$, $x_3 = P(B \cap (A \cup C)^c)$, $x_4 = P(A \cap C \cap B^c)$, $x_5 = P(A \cap B \cap C)$, $x_6 = P(B \cap C \cap A^c)$, $x_7 = P(C \cap (A \cup B)^c)$.



Then, using the information given, we have

$$\begin{aligned}
P(A) &= x_1 + x_2 + x_4 + x_5 = 0.4 &(1)\\
P(B) &= x_2 + x_3 + x_5 + x_6 = 0.7 &(2)\\
P(A \cup B) &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0.8 &(3)\\
P(C \cap (A \cup B)) &= x_4 + x_5 + x_6 = 0.2 &(4)\\
P(B \cap (A \cup C)) &= x_2 + x_5 + x_6 = 0.4 &(5)\\
P(C) &= x_4 + x_5 + x_6 + x_7 = 0.3 &(6)\\
P(B \cap C) &= x_5 + x_6 = 0.2 &(7)
\end{aligned}$$

Using (1)-(3), we get

$$P(A \cap B) = x_2 + x_5 = 0.3 \qquad (8)$$

Using (8) and (5), we get $x_6 = 0.1$. Using (4) and (7), we get $x_4 = 0$. Putting these in (4), we get $x_5 = 0.1$. So, (6) yields $x_7 = 0.1$, and (5) yields $x_2 = 0.2$. From (1) and (2), we now get $x_1 = 0.1$ and $x_3 = 0.3$.

(a) The probability that exactly two of A, B and C occur equals $x_2 + x_4 + x_6 = 0.3$.

(b) The probability that none of A, B and C occur equals $1 - \sum_{k=1}^{7} x_k = 0.1$.

(c) The probability that A and exactly one of B and C occur equals $x_2 + x_4 = 0.2$.

Assignment 2. Denote by 'H' and 'T' the events that a head and a tail occurs, respectively. A sample point $\{x, y, z\}$ with $x, y, z \in \{H, T\}$ will denote that x occurs for A, y occurs for B and z occurs for C.

(a) $\Omega = \{\{H,H,H\}, \{H,H,T\}, \{H,T,H\}, \{H,T,T\}, \{T,H,H\}, \{T,H,T\}, \{T,T,H\}, \{T,T,T\}\}$.

(b) Assuming the each coin is fair, all the sample points may be assigned equal probabilities, i.e., $P(\{x, y, z\}) = 1/8$ for all $\{x, y, z\} \in \Omega$.

(c) $\mathrm{P}(A \text{ wins on first toss}) = \mathrm{P}(\{H,T,T\} \cup \{T,H,H\}) = 1/4$. The probabilities are the same

for B as well as C winning on the first toss.
(d) P(no winner on first toss) = $\mathrm{P}(\{H,H,H\} \cup \{T,T,T\}) = 1/4$.
(e) Let F and G denote the events that A wins on first toss and that the winner is decided on first toss, respectively. Then, $\mathrm{P}(G) = \mathrm{P}(A$ wins on first toss) + P(B wins on first toss) + P(C wins on first toss) = 3/4. Since $F \subset G$, we have $\mathrm{P}(F \cap G) = \mathrm{P}(F) = 1/4$. So, $\mathrm{P}(F \mid G)$ $= \mathrm{P}(F \cap G)/\mathrm{P}(G) = 1/3$.
(Intuition : since all events are equally likely, if there is a winner on the first toss, then it has to be one of A, B and C with equal probability. This probability is obviously $1/3$).

Assignment 3. Since only one prisoner will be pardoned and that too at random, P(A will be pardoned) = P(B will be pardoned) = P(C will be pardoned) = 1/3.
However, as per the understanding between prisoner A and the jailer, the following are true :
(i) P(A is told that C will be executed | A will be pardoned) = P(A is told that B will be executed | A will be pardoned) = 1/2 (assuming that the jailer chooses between B and C at random),
(ii) P(A is told that C will be executed | B will be pardoned) = 1 (since the jailer is left with no other choice and he cannot tell A his fate),
(iii) P(A is told that C will be executed | C will be pardoned) = 0 (assuming that the jailer does not lie).
Thus, by the law of total probability,
P(A is told that C will be executed)
= P(A will be pardoned) × P(A is told that C will be executed | A will be pardoned)
+ P(B will be pardoned) × P(A is told that C will be executed | B will be pardoned)
+ P(C will be pardoned) × P(A is told that C will be executed | C will be pardoned)
$= (1/3) \times (1/2) + (1/3) \times 1 + (1/3) \times 0 = 1/2$.
(a) By Bayes' theorem,
P(A will be pardoned | A is told that C will be executed)
= P(A will be pardoned ∩ A is told that C will be executed)/P(A is told that C will be executed)
= P(A will be pardoned) × P(A is told that C will be executed | A will be pardoned)/(1/2)
$= (1/3) \times (1/2)/(1/2) = 1/3$.
(b) By Bayes' theorem,
P(B will be pardoned | A is told that C will be executed)
= P(B will be pardoned ∩ A is told that C will be executed)/P(A is told that C will be executed)
= P(B will be pardoned) × P(A is told that C will be executed | B will be pardoned)/(1/2)
$= (1/3) \times 1/(1/2) = 2/3$.
So, prisoner A was wrong in his assessment that the jailer's information will increase his chances of getting a pardon. In fact, the remaining prisoner is the one who benefits from this information (provided of course that A shares this information with him !).
(c) Using similar argument as above, we have P(A_j will be pardoned) = $1/n$ for $j = 1, 2, \ldots, n$, and
(i) P(A_1 is told that $A_3, A_4, \ldots, A_n$ will be executed | A_1 is pardoned) = $1/(n-1)$ (assuming that the jailer chooses randomly among one of $(n-1)$ other prisoners to exclude from his list that he will tell A_1),
(ii) P(A_1 is told that $A_3, A_4, \ldots, A_n$ will be executed | A_2 is pardoned) = 1 (since the jailer has no other option in this case),

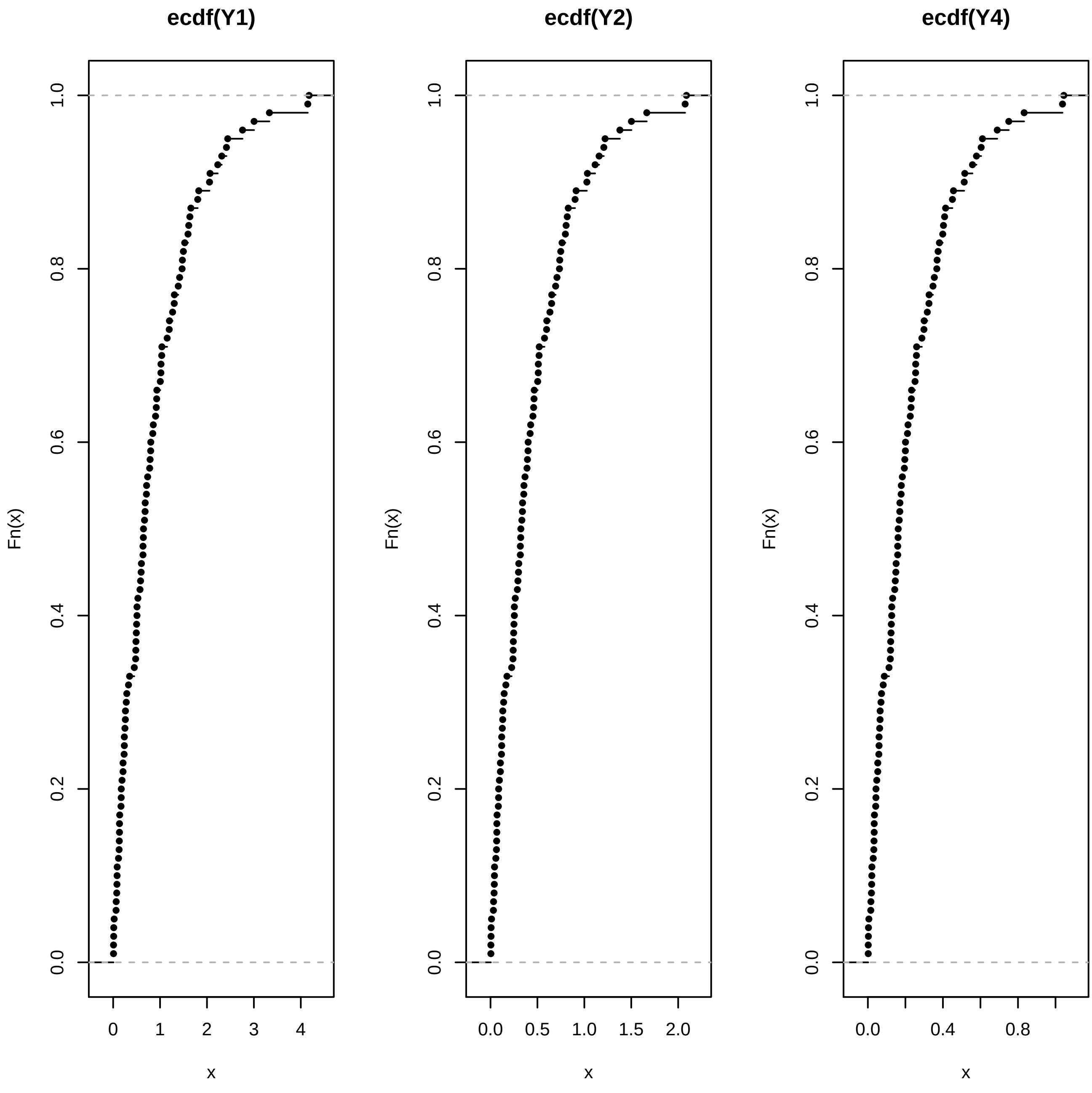
(iii) $\mathrm{P}(A_1$ is told that $A_3, A_4, \ldots, A_n$ will be executed $\mid A_k$ is pardoned$) = 0$ for all $k = 3, 4, \ldots, n$ (assuming that the jailer does not lie).
Thus, by the law of total probability,
$\mathrm{P}(A$ is told that $A_3, A_4, \ldots, A_n$ will be executed$)$
$= (1/n) \times [1/(n-1)] + (1/n) \times 1 + \sum_{k=3}^{n} (1/n) \times 0 = 1/(n-1)$.
So, by Bayes' theorem, $\mathrm{P}(A_1$ will be pardoned $\mid A$ is told that $A_3, A_4, \ldots, A_n$ will be executed$)$
$= (1/n) \times [1/(n-1)]/[1/(n-1)] = 1/n$.
On the other hand, $\mathrm{P}(A_2$ will be pardoned $\mid A$ is told that $A_3, A_4, \ldots, A_n$ will be executed$)$
$= (1/n) \times 1/[1/(n-1)] = (n-1)/n$.

Assignment 4. Denote by Pos and Neg the events that the test result is positive and negative, respectively. Also, denote by D and ND the events that the person actually has the disease and does not have it, respectively. We need to find $\mathrm{P}(D \mid Pos)$. It is given that $\mathrm{P}(Pos \mid ND) = 1/100$, $\mathrm{P}(Neg \mid D) = 2/100$ and $\mathrm{P}(D) = 1/1000$.
Now, $\mathrm{P}(Pos) = \mathrm{P}(Pos \cap D) + \mathrm{P}(Pos \cap ND)$
$= \mathrm{P}(D)$ - $\mathrm{P}(Neg \cap D) + \mathrm{P}(Pos \cap ND)$
$= \mathrm{P}(D)$ - $\mathrm{P}(D) \times \mathrm{P}(Neg \mid D) + \mathrm{P}(ND) \times \mathrm{P}(Pos \mid ND)$
$= (1/1000) - (1/1000) \times (2/100) + (999/1000) \times (1/100) = 1097/(1000 \times 100) \approx 11/1000$.
The above calculations show that $\mathrm{P}(Pos \cap D) = (1/1000) - (1/1000) \times (2/100) = 98/(1000 \times 100) \approx 1/1000$.
So, we have $\mathrm{P}(D \mid Pos) = \mathrm{P}(Pos \cap D)/\mathrm{P}(Pos) = 98/1097 \approx 1/11$.
Further, $\mathrm{P}(ND \mid Neg) = \mathrm{P}(Neg \cap ND)/\mathrm{P}(Neg)$
$= [\mathrm{P}(ND)$ - $\mathrm{P}(Pos \cap ND)]/[1$ - $\mathrm{P}(Pos)]$
$= [(999/1000) - (999/1000) \times (1/100)]/[1 - 1097/(1000 \times 100)]$
$= 98901/98903 \approx 1$.
Thus, although the chances of false positives (namely, $\mathrm{P}(Pos \mid ND)$) and true negatives (namely, $\mathrm{P}(Neg \mid D)$) from the test procedure are very small, it is not a good idea to base an affirmative diagnosis of the disease on this test alone.

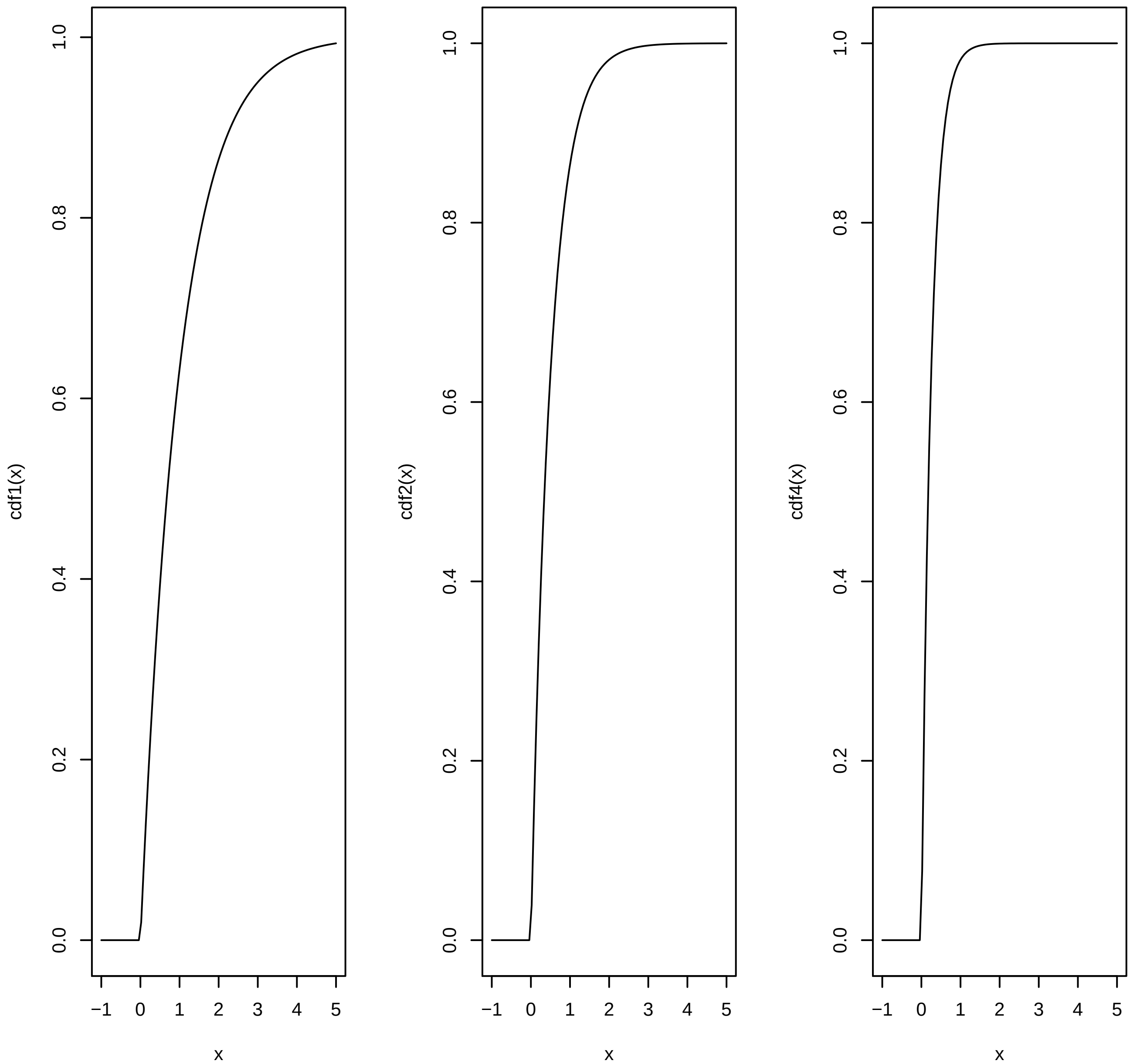
Assignment 5. (e) These plots show the empirical distribution functions

$$t \mapsto \frac{1}{100} \sum_{i=1}^{100} \mathbf{1}\{Y_i \leq t\}$$

for each of the vectors $Y1$, $Y2$ and $Y4$. It is a nondecreasing right-continuous step function (piecewise constant) that equals 0 for t (strictly) smaller than the minimum of the sample, and 1 for t (weakly) larger than the maximum of the sample.



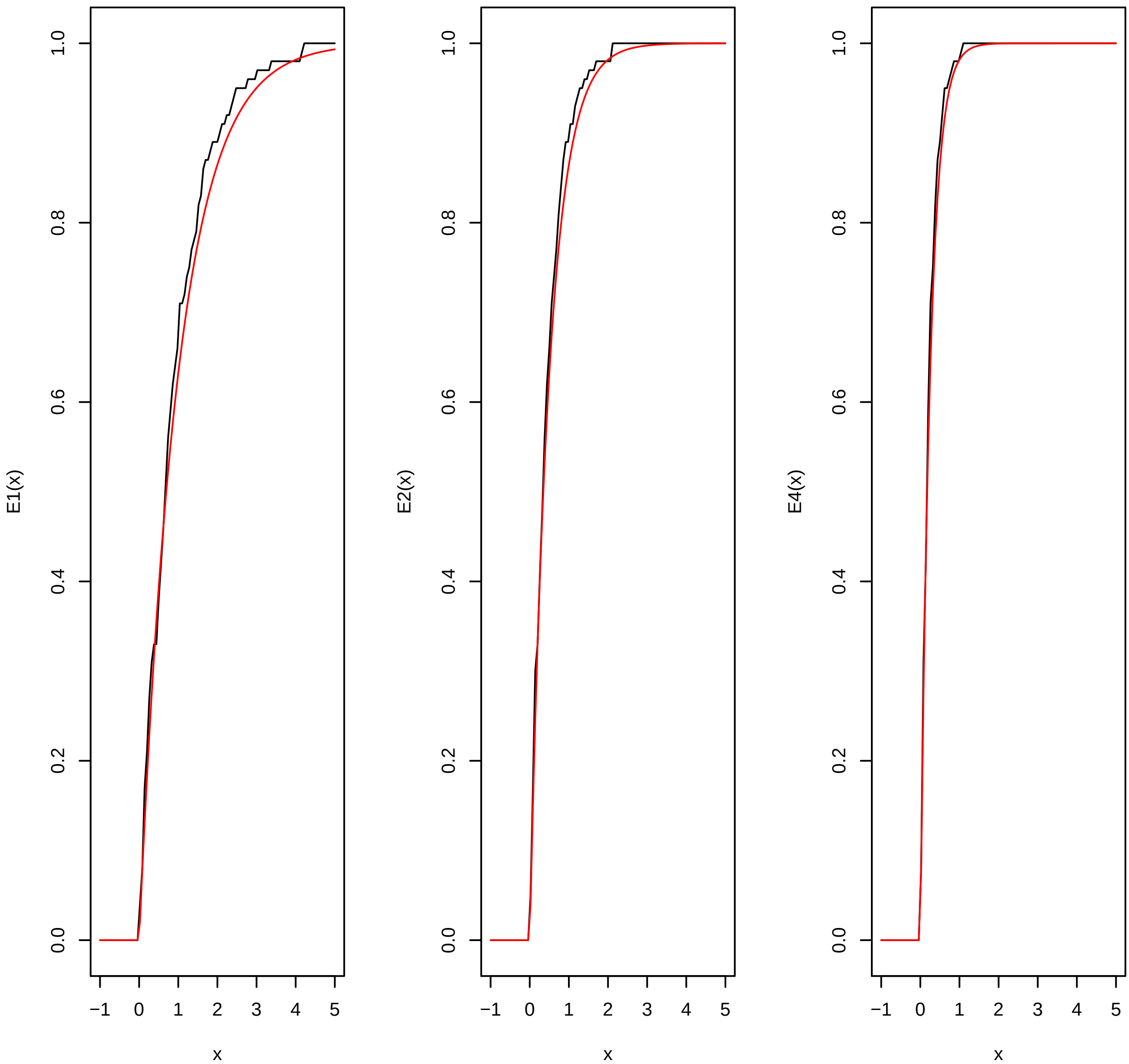
(g) These are plots of the functions F_λ for the parameter values 1, 2, and 4 for λ.



(i) These figures combine the two previous plots. We see that the empirical distribution function for $Y\lambda$ is very close to the function F_λ. The figures imply that $Y\lambda$ follows the distribution F_λ, i.e. Exp(λ). And indeed, since $X \sim$Unif(0,1),

$$P(Y\lambda \leq t) = P(-\log(1-X)/\lambda \leq t) = P(X \leq 1 - \exp(-\lambda t)) = F_\lambda(t).$$

The fact that the empirical distribution function approaches the distribution function itself is known as the *Glivenko–Cantelli* theorem.



(j) q_λ is the inverse of F_λ on $(0, \infty)$.
(k) Yes. This is so because $Y = q(X) = F^{-}(X) = F^{-1}(X)$ is such that

$$P(Y \leq t) = P(X \leq F(t)) = F(t).$$