Assignment sheet 9

November 22, 2017

Assignment 1. Prove deterministic version of the optimal dimension reduction of slide 297.

Assignment 2. (a) Let $\Omega_{n\times n}$ be a strictly positive definite matrix, and let $A_{p\times n}$ be of rank $p \leq n$ (linearly indpendent columns). Show that $B_{p\times p} = A^T\Omega A$ is strictly positive definite, hence invertible. Deduce that A^TA is strictly positive definite and invertible.
(b) Show by a counter example that if Ω is only assumed symmetric and invertible, then B is not necessarily invertible. *Hint : the simplest example for* $\Omega_{2\times 2}$ *will work here.*

Assignment 3. Let $X = (X_1, \dots, X_k)$ be a random vector with finite variance and independent coordinates X_i. We know that

$$X_i \sim N(0, \sigma^2) \text{ for all } i \Longrightarrow c^TX \text{ has the same distribution for any } c \in \mathbb{S}^{k-1}$$

where $\mathbb{S}^{k-1} = \{c \in \mathbb{R}^k : \|c\| = 1\}$.
We shall see in this assignement that the converse is true. Assume $k = 2$ (the proof for general k is the same).
Let $V = (X, Y)^T$ be a random vector in $\mathbb{R}^2$, such that

(i). X and Y are independent ;
(ii). $\mathbb{E}X^2 = \sigma^2 < \infty$,
(iii). c^TV has the same distribution for all $c \in \mathbb{S}^1$.

In the following $X_1, X_2, \dots$ represent independent copies of X.
By a judicious choice of $c \in \mathbb{S}^1$,

(i) show that $c^TV \sim X$, for all $c \in \mathbb{S}^1$.
(ii) show that X and Y have the same distribution.
(iii) Find the expectation of X.
(iv) Show that $X \sim \frac{1}{\sqrt{2}}(X_1 + X_2)$.
(v) By induction show that the distribution of $n^{-1/2}\sum_{i\leq n} X_i$ is the same for all n.
(vi) Use the central limit theorem to conclude that $X \sim N(0, \sigma^2)$, and therefore $Y \sim N(0, \sigma^2)$ as well.

Remark. The result is no longer true if X and Y are not assumed independent.

> **Remark :** If X is $n \times p$ with $p \leq n$, then
>
> X full rank $\Leftrightarrow$ X is injective $\Leftrightarrow$ the coloumns of X are lin. independent.
>
> Note that the first implication is false when $p > n$.

Assignment 4 (Non linear $\leftrightarrow$ linear)**.** This exercise aims to show that a non linear model could, sometimes, be transformed into a linear one.
For example, the model $y = \beta_1(x + \beta_3)^{\beta_2}(\varepsilon^2 + 1)$ can be written as

$$\log(y) = \underbrace{\log(\beta_1)}_{\beta_1^*} + \underbrace{\beta_2}_{\beta_2^*} \log(x + \beta_3) + \underbrace{\log(\varepsilon^2 + 1)}_{\varepsilon^*},$$

with $\begin{bmatrix}1 & \log(x+\beta_3)\end{bmatrix}$ as design matrix. Moreover, in order for the transformation to be possible it must be $\beta_1 > 0, x + \beta_3 > 0$.
Write, whenever is possible, the following model as linear regression. Specify each time the new parameters β^*, the new error term ε^*, the restrictions (e.g. $\beta_1 > 0$) and give the design matrix, as in the previous example :

a) $y = \beta_0 + \beta_1/x + \beta_2/x^2 + \varepsilon$
b) $y = \beta_0/(1 + \beta_1 x) + \varepsilon$
c) $y = \beta_0/(\beta_1 x) + \varepsilon$
d) $y = 1/(\beta_0 + \beta_1 x + \varepsilon)$
e) $y = \beta_0 + \beta_1 x_1^{\beta_2} + \beta_3 x_2^{\beta_4} + \varepsilon$
f) $y = \beta_1 x_1^{\beta_2} \cos(x_2)^{\beta_3} \varepsilon$
g) $y = \beta_1 + x_1^{\beta_2}(2 + \cos(x_2))^{\beta_3}(\varepsilon^2 + 1)$
h) $y = \beta_0 + \beta_1 \cos(x + \beta_2) + \varepsilon$

Assignment 5. In `R` we can write a linea model through the following command

`reponse~expression,`

where `reponse` might sometimes be absent and `expression` is a collection of terms joined by operators, all normally assembled into an arithmetic expression. For example, suppose that

$$y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 1 \\ 5 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix},$$

and let `x`, `a`, `b` the columns of $X = [x, a, b]$.
Here some `R` commands and the model that they represent :

Command `R`	Model
`y~x`	$y_j = \beta_0 + \beta_1 x_j + \varepsilon_j$
`y~x-1`	$y_j = \beta_1 x_j + \varepsilon_j$
`y~x+a`	$y_j = \beta_0 + \beta_1 x_j + \beta_2 a_j + \varepsilon_j$.

Write down the design matrix corresponding to the formula : (i) `y~a-1`, (ii) `y~a+b`.

Assignment 6. We are given the weights of two groups of rats at the beginning and at the end of a 15-days long experiment. During these 15 days, a group is fed normally, the other with some growth inibitors. Let x the weight at the beginning of the experiment and y the weight at the end. Assume the following linear model for the weights :

$$y_{jg} = \alpha_g + \beta_g x_{jg} + \varepsilon_{jg}; \qquad j = 1, 2, 3; \quad g = 1, 2.$$

a) Write the design matrix and the vectors of parameters.
b) We are interested in the following models : (i) $\beta_1 = \beta_2$ (if plotting, we will get two parallel lines), (ii) $\alpha_1 = \alpha_2$ (if plotting, we would get the same intercept at the origin) et (iii) $\alpha_1 = \alpha_2$ et $\beta_1 = \beta_2$ (same line for the two groups).
Find a model that admits (i), (ii), (iii) as sub-models (a sub-model is obtained by a full model by fixing some parameters, in this case to zero). Give the design matrix of this

model as well as the parameter vectors. Indicate as well which columns of the design matrix one needs to suppress in order to retain each one of the sub-models.
Hint : To write the model, some parameters needs to be set equal to 0. Write the model by using $\alpha_2 - \alpha_1$ and $\beta_2 - \beta_1$ as parameters.

Assignment 7. Let $X = QR = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$ the QR decomposition for a full-rank matrix $X_{n\times p}$ where $n \geq p$. Assume that y is a vector of n observations sampled from $Y \sim N_n(X\beta, \sigma^2 I)$.

a) Write $Z = Q^{\mathrm{T}} Y$. Show that $Z \sim N_n(R\beta, \sigma^2 I)$.

b) Let $u = Q^{\mathrm{T}} \hat{y}$. Show that $u = \begin{pmatrix} Q_1^{\mathrm{T}} y \\ 0 \end{pmatrix}$.

c) Let $v = Q^{\mathrm{T}} e$. Show that $v = \begin{pmatrix} 0 \\ Q_2^{\mathrm{T}} y \end{pmatrix}$.

d) Using a)–c), show that $\hat{\beta}$ and S^2 are independent. *(Hint : $z = u + v$).*

> QR **decomposition :** All real matrices $X \in \mathbb{R}^{n\times p}$, with $n \geq p$, admit a decomposition
> $$X_{n\times p} = Q_{n\times n} R_{n\times p},$$
> where Q is an orthogonal matrix and R is an upper triangular matrix ($R_{ij} = 0$ si $i > j$). Sometimes it is useful to write
> $$X = QR = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = Q_1 R_1,$$
> where Q_1 is $n \times p$, and R_1 is $p \times p$. One can show that for each matrix X of full rank it is possible to chose $R_{1,ii} > 0$, $i = 1, \ldots, p$, and that the decomposition $X = Q_1 R_1$ is unique.