Assignment sheet 8

November 15, 2017

Assignment 1. (a) Let $A_{n\times m}$ and $B_{m\times n}$ be matrices such that AB is square. Show that $tr[AB] = tr[BA]$.
(b) Suppose that ABC is well-defined as a square matrix. Show that $tr[ABC] = tr[BCA]$.
Warning : in general ACB is not even defined, but even if it is, usually $tr[ACB] \neq tr[ABC]$!
(c) Let A be a random matrix (each of its coordinates is a random variable). Show that $\mathbb{E}(tr[A]) = tr[\mathbb{E}(A)]$, where the last expectation is interpreted coordinatewise.

Assignment 2. (288) Let Q be a projection and λ an eigenvalue of Q. Show that $\lambda = 0$ or 1.

Assignment 3. (289) (a) Let P be a projection on a subspace V. Show that $Pv = v$ for all $v \in V$.
(b) Show that $Pw = 0$ for all $w \in V^{\perp}$. *Hint : compute $(Pw)^T x$ for $w \in V^{\perp}$ and $x \in \mathbb{R}^p$.*
(c) Let Q be another projection on the same space V. Show that $P = Q$.

Assignment 4. Show that the matrices P and Q below are non-orthogonal projections ($P = P^2$, $Q = Q^2$) on the same subspace V :

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Assignment 5. (289) Let $x_1, \ldots, x_p$ be linearly independent vectors in $\mathbb{R}^n$ and X be an $n \times p$ matrix with columns $x_1, \ldots, x_p$. Show that :
(a) the subspace $V = span(x_1, \ldots, x_p)$ equals $M(X)$.
(b) $X^T X$ is invertible. *Hint : take v in the kernel and compute $\|Xv\|^2$.*
(c) the projection onto V is

$$H = X(X^T X)^{-1} X^T.$$

Assignment 6. The formula for H seems a bit magical, and here we shall see how to get to it. Let $x_1, \ldots, x_p$ be vectors in $\mathbb{R}^n$, and suppose that we wish to find the projection H onto their span V. Let X be a matrix with columns $x_1, \ldots, x_p$.
(a) Explain why we can assume without loss of generality that (x_j) are independent.
(b) Explain why it makes sense to guess that H should take the form XM for some matrix M.
(c) Explain why it makes sense to guess that H should take the form NX^T for some matrix N.
(d) In view of (b) and (c), write $H = XBX^T$ for some $p \times p$ matrix B. Find B. *Hint : let $e_i \in \mathbb{R}^p$ be the i-th unit vector, then $x_i = Xe_i$.*

Assignment 7. (293) show that the two definitions of a positive definite matrix are equivalent.

Assignment 8. (295) Show that Q is an orthogonal projection of rank k if and only if there exist orthonormal vectors $v_1, \ldots, v_k$ such that $Q = \sum_{i=1}^{k} v_i v_i^T$.

Assignment 9. (a) Let $Z \sim N(0_p, I_{p\times p})$ and H be a projection of rank r. Show that $Z^T H Z \sim \chi^2_r$. *Hint : use the spectral decomposition of H.*
(b) Let $Y \sim N(\mu_p, \Omega_{p\times p})$ with Ω nonsingular. Show that $(Y - \mu)^T \Omega^{-1} (Y - \mu) \sim \chi^2_p$. *Hint : reduce to (a).*