Assignment sheet 3

October 4, 2017

Assignment 1. Prove that the following distributions are members of an exponential families by finding their natural and their usual parametrisation.
(i) The Poisson distribution.
(ii) The Geometric distribution.
(iii) The exponential distribution.
(iv) The Gamma distribution.

Assignment 2. Prove that the multinomial distribution (with K known) is a member of an exponential family.

Assignment 3. Using the factorisation theorem, find a sufficient statistics for an independent sample $Y = (Y_1, \ldots, Y_n)$ from the following distributions :
(i) The Poisson distribution.
(ii) The Geometric distribution.
(iii) The exponential distribution.
(iv) The Gamma distribution with parameters (λ, r) for λ assumed to be a known number.
You can check your results using your knowledge from Exponential Families.

Assignment 4. In this assignment, we will find the distribution of the maximum and the minimum *order statistics* for a sample.
Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a probability distribution with c.d.f. F and density function f. Let $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$.
(a) Find the c.d.f.'s of $X_{(1)}$. Hence find the density function of $X_{(1)}$.
(*Hint :* $\mathbb{P}[X_{(1)} > y] = \mathbb{P}[X_1 > y, X_2 > y, \ldots, X_n > y]$ *since the minimum is larger than* y *iff all sample values are larger than* y*. Now compute* $\mathbb{P}[X_{(1)} \leq y]$ *using the formula* $1 - \mathbb{P}[X_{(1)} > y]$.)
(b) Find the c.d.f.'s of $X_{(n)}$. Hence find the density function of $X_{(n)}$.
(*Hint :* $\mathbb{P}[X_{(n)} \leq z] = \mathbb{P}[X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z]$ *since the maximum is smaller than or equal to* z *iff all sample values are smaller than or equal to* z.)
(c) Find the joint c.d.f. of $\mathbf{W} = (X_{(1)}, X_{(n)})^\top$.
(*Hint : Fist compute* $\mathbb{P}[X_{(1)} > y, X_{(n)} \leq z]$ *using arguments as in (a) and (b). Then, compute* $\mathbb{P}[X_{(1)} \leq y, X_{(n)} \leq z]$ *using the formula* $\mathbb{P}[X_{(n)} \leq z] - \mathbb{P}[X_{(1)} > y, X_{(n)} \leq z]$*. Be CAREFUL about the ranges of the variables* y *and* z.)
(d) Use part (c) to find the joint density function of $\mathbf{W} = (X_{(1)}, X_{(n)})^\top$. Are $X_{(1)}$ and $X_{(n)}$ independent ?
(e) Compute the marginal c.d.f.s and density functions of $X_{(1)}$ and $X_{(n)}$ for the Unif(0,θ) distribution. Also find the joint density function of $X_{(1)}$ and $X_{(n)}$.
(f) Discuss the behaviour of the marginal c.d.f. of $X_{(n)}$ in part (e) when $n \to \infty$.
(*Hint : Write the c.d.f. for* z *ranging on the whole of* $\mathbb{R}$ *and then find the limit. Be CAREFUL at the point* $z = \theta$.)
(g) Repeat part (e) for the distribution with density function $f(x) = \exp\{-(x - \lambda)\}, x \geq \lambda$, and $f(x) = 0$, otherwise, where $\lambda > 0$.
(Note : The above distribution is that of an Exp(1) conditional on the event $[X \geq \lambda]$.)
(h) Discuss the behaviour of the marginal c.d.f. of $X_{(1)}$ in part (g) when $n \to \infty$.
(*Hint : Write the c.d.f. for* y *ranging on the whole of* $\mathbb{R}$ *and then find the limit. Be CAREFUL at the point* $y = \lambda$.)

Assignment 5. Let V be a $N(\mu, \sigma^2)$ variable.
(a) Find the c.d.f. and the density function of V^2 directly by computing $\mathbb{P}[V^2 \leq w]$.
(Note : This is an example of a non-monotone transformation of variable, and we solve it directly.)
(b) In case $\mu = 0$ and $\sigma^2 = 1$, can you identify the distribution obtained in (a) ?
(c) Let U be a $N(\theta, \tau^2)$ variable that is independent of V. Prove that U and V^2 are independent by showing that

$$\mathbb{P}[U \leq u, V^2 \leq w] = \mathbb{P}[U \leq u] \ \mathbb{P}[V^2 \leq w].$$

(Note : It can be proved that if U and V are independent, then so are $f(U)$ and $g(V)$, where f and g are two "measurable" functions.)
Recall Problem (1) in Week 2 :
Let X_1, X_2 be i.i.d. $N(\gamma, \eta^2)$ variables. Let $\overline{X}$ and S^2 be the sample mean and the sample variance.
(d) Show that $S^2 = (X_1 - X_2)^2/2$.
(e) Show using part (c) that $\overline{X}$ and S^2 are independent.
(Note : It can be shown that even for $n > 2$, we have the independence of $\overline{X}$ and S^2.)
(f) In case $\gamma = 0$, what is the distribution of $T = (X_1 + X_2)/|X_1 - X_2|$?
(*Hint : Try to relate T to $\overline{X}$ and S^2.*)
(g) Run the following command :

```
vals = matrix(0,nrow=1000,ncol=2)
n = 100
for (i in 1:1000)
{
set.seed(i+04102017)
X = rnorm(n)
vals[i,1] = mean(X)
vals[i,2] = var(X)
}
plot(vals[,1],vals[,2],xlab=expression(bar(X)),ylab=expression(S^2))
abline(v=0,col="red")
abline(h=1,col="red")
```

Try to guess the value of $Cov(\overline{X}, S^2)$ from the plot. Justify your answer.

Assignment 6. In this assignment we shall see how the entropy is related to information. Let X be a *discrete* random variable.
(a) Show that $H(X) \geq 0$. *Note : $H(X)$ can be infinite, but this happens for rather pathological distributions.*
(b) Let g be an injective function. Show that $H(g(X)) = H(X)$. What this means is that the entropy does not see what *values* X takes, but only their *probabilities.*
(c) Let X take the values $-1, 0, 1$ with probabilities $0 < p_1, p_2, p_3$, $p_1 + p_2 + p_3 = 1$. Show that $H(X^2) < H(X)$. The entropy decreased because we lost information about X : we only know its absolute value but not the sign.
Hint : the function $h(x) = x \log x$ is superadditive *: $h(x+y) > h(x) + h(y)$ if $x, y > 0$.*
Remark : the result holds more generally : if g is not injective (and $H(X)$ is finite), then $H(g(X)) < H(X)$.

(d) The situation in the continuous case is far from being obvious. Suppose that $X \sim$Unif$[0, \theta]$ for $\theta > 0$. Find $H(X)$.
(e) Is it true that $H(X) \geq 0$?
(f) Is it true that $H(g(X)) \leq H(X)$? What if g is injective ? *Hint : take* $\theta = 1$ *and* g *linear.*

Assignment 7. In this assignment we shall see how the t distribution behaves when the number of degrees of freedom changes.
(a) Use the commands

```
set.seed(04102017)
N <- 1e5
rt(N, df = 1)
```

to generate a sample of size $100'000$ from a t distribution with one degree of freedom. Store the values as a vector X.
(b) Use the command

```
sum(-1 < X & 1 > X)
```

to compute how many elements of X are in $(-1, 1)$. Modify the command to obtain the *proportion* of elements of X that are in $(-1, 1)$.
(c) Repeat this for $2, 4, 12, 25, 100, 250, 500$ degrees of freedom. Store the resulting proportions in a vector `small` of length 8. *Hint : you can do this in an automated way with the following commands.*

```
df <- c(1, 2, 4, 12, 25, 100, 250, 500)
small <- numeric(length(df))
```

Then use a for loop that will set each element in `small` using the commands in (a) and (b).
Remark. The use of `numeric(length(df))` instead of simply putting 8 makes the code more easily modifiable : if you change the length of the vector `df` then the length of `small` adapts automatically.
(d) Do the same for a random variable $N(0, 1)$, and store the proportion as a variable `small.norm`.
(e) Plot the values of `small` as a function of the degrees of freedom. Compare with the value of `small.norm`. What do you observe ? *Hint : after you draw the plot you can use the following command (modify it appropriately) :*

```
abline(h = 0.63)
```

(f) Use the commands

```
xval <- seq(from = -6, to = 6, length.out = 1000)
plot(xval, dnorm(xval), type = "l", xlab = "", ylab = "")
lines(xval, dt(xval, df = 1))
```

To simultaneously plot the densities of the $N(0, 1)$ and t_1 distributions. Use an appropriate `for` loop for plotting the densities corresponding to the different degrees of freedom in (c). What do you observe ? *Hint : you can add* `col = i` *to the command* `lines` *so that different degrees of freedom will have different colours. It may be helpful to redraw the normal density again after the for loop.*